Unit III

Problems on unconstrained Multivariate Optimization: NLPP - Non Linear Programming Problem

Find the extreme point and its nature of the function. $f(x_1, x_2) = x_1^3 + x_2^3 + 2x_1^2 + 4x_2^2 +$

Necessary Condition:

$$\frac{\partial f}{\partial x_1} = 0 \Rightarrow 3x_1^2 + 0 + 4x_1 + 0 + 0 = 0$$

$$\Rightarrow 3x_1^2 + 4x_1 = 0$$

$$\Rightarrow x_1(3x_1 + 4) = 0$$

$$\Rightarrow x_1 = 0, \quad 3x_1 + 4 = 0$$

$$3x_1 = -4$$

$$\Rightarrow x_1 = 0 \qquad x_1 = -4/3$$

$$\frac{\partial f}{\partial x_2} = 0 \Rightarrow 3x_2^2 + 8x_2 = 0$$

$$\Rightarrow x_2(3x_2 + 8) = 0$$

$$\Rightarrow x_2 = 0, \quad x_2 = -8/3$$

Extreme points are

$(0, 0)$, $(0, -8/3)$ $(-4/3, 0)$, $(-4/3, -8/3)$

Hermitian matrix

$$H = \begin{pmatrix} \frac{\partial^2 f}{\partial x_1^2} & \frac{\partial^2 f}{\partial x_1 \partial x_2} \\ \frac{\partial^2 f}{\partial x_2 \partial x_1} & \frac{\partial^2 f}{\partial x_2^2} \end{pmatrix}$$

$$\frac{\partial^2 f}{\partial x_1^2} = 6x_1 + 4.$$

$$\frac{\partial^2 f}{\partial x_1 \partial x_2} = \frac{\partial}{\partial x_1}\left(3x_2^2 + 8x_2\right) = 0$$

$$\frac{\partial^2 f}{\partial x_2 \partial x_1} = \frac{\partial}{\partial x_2}\left(\frac{\partial f}{\partial x_1}\right) = \frac{\partial}{\partial x_1}\left(3x_1^2 + 4x_1\right) = 0$$

$$\frac{\partial^2 f}{\partial x_2^2} = 6x_2 + 8$$

$$H = \begin{pmatrix} 6x_1 + 4 & 0 \\ 0 & 6x_2 + 8 \end{pmatrix}$$

Case(i)

At $(0, 0)$

$$H = \begin{pmatrix} 4 & 0 \\ 0 & 8 \end{pmatrix}.$$

$$|H_1| = |4| = 4 > 0$$

$$|H_2| = \begin{vmatrix} 4 & 0 \\ 0 & 8 \end{vmatrix} = 32 > 0$$

All the Principal minors are > 0. $(0, 0)$ is called Relative minimum.

called $\boxed{H \text{ is positive definite}}$

Case(ii): At $(0, -8/3)$

$$H = \begin{pmatrix} 4 & 0 \\ 0 & 6\left(\frac{-8}{3}\right) + 8 \end{pmatrix}$$

$$H = \begin{pmatrix} 4 & 0 \\ 0 & -8 \end{pmatrix} \Rightarrow |H_1| = |4| > 0$$

$$|H_2| = \begin{vmatrix} 4 & 0 \\ 0 & -8 \end{vmatrix} = -32 < 0$$

H is indefinite and ~~At~~ The point $(0, -8/3)$ is neither relative minimum (nor) relative maximum.

Case(iii)

At $\left(-\frac{4}{3}, 0\right)$

$$H = \begin{pmatrix} 6\left(-\frac{4}{3}\right)+4 & 0 \\ 0 & 6(0)+8 \end{pmatrix} = \begin{pmatrix} -8+4 & 0 \\ 0 & 8 \end{pmatrix} = \begin{pmatrix} -4 & 0 \\ 0 & 8 \end{pmatrix}$$

$|H_1| = |-4| = -4 < 0$

$|H_2| = \begin{vmatrix} -4 & 0 \\ 0 & 8 \end{vmatrix} = -32 < 0$

H is indefinite. and $(-4/3, 0)$ is neither relative minimum nor relative maximum.

Case(iv)

At $\left(-\frac{4}{3}, -\frac{8}{3}\right)$

$$H = \begin{pmatrix} 6\left(-\frac{4}{3}\right)+4 & 0 \\ 0 & 6\left(-\frac{8}{3}\right)+8 \end{pmatrix}$$

$$= \begin{pmatrix} -8+4 & 0 \\ 0 & -8 \end{pmatrix} = \begin{pmatrix} -4 & 0 \\ 0 & -8 \end{pmatrix}$$

$|H_1| = |-4| = -4 < 0$

$|H_2| = \begin{vmatrix} -4 & 0 \\ 0 & -8 \end{vmatrix} = 32 - 0 = 32 > 0$ } -ve to +ve

H is negative definite.

and $\left(-\frac{4}{3}, -\frac{8}{3}\right)$ is relative maximum.

xxxxx

Problem 2:

If $u = -x^3 + 3xz + 2y - y^2 - 3z^2$.

Find maximum or minimum values.

Solution:

$u = -x^3 + 3xz + 2y - y^2 - 3z^2$

$\frac{\partial u}{\partial x} = -3x^2 + 3(1)z + 0 - 0 - 0$

$= -3x^2 + 3z$

$\frac{\partial u}{\partial y} = 0 + 0 + 2(1) - 2y^2 - 0$

$= 2 - 2y$

$\frac{\partial u}{\partial z} = 0 + 3x(1) + 0 - 0 - 3(2z)$

$= 3x - 6z$

$u_x = 0 \Rightarrow -3x^2 + 3z = 0 \rightarrow (1)$

$u_y = 0 \Rightarrow 2 - 2y = 0$

$2y = 2$

$\boxed{y = 1}$

$u_z = 0 \Rightarrow 3x - 6z = 0 \rightarrow (2)$

From equation (1), $-3x^2 + 3z = 0 \rightarrow (1)$

From equation (2) $3x - 6z = 0 \rightarrow (2)$

$(1) \times 2 \Rightarrow -6x^2 + 6z = 0$

(+) $3x - 6z = 0$

(2) $-6x^2 + 3x = 0$

$3x - 6x^2 = 0$

$3x(1 - 2x) = 0$

$\Rightarrow x = 0, \quad 1 - 2x = 0$

$\Rightarrow x = 0, \quad x = 1/2$

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When $x = 1/2$

$$3x - 6z = 0$$

$$\Rightarrow 3(1/2) - 6z = 0$$

$$\Rightarrow \frac{3}{2} - 6z = 0$$

$$\Rightarrow 6z = \frac{3}{2}$$

$$\Rightarrow z = \frac{3}{12} = \frac{1}{4}.$$

$$\Rightarrow \boxed{z = 1/4}$$

$\therefore$ Critical Points are

$$\boxed{x = 1/2, \quad y = 1, \quad z = 1/4}$$

Construction of Hessian Matrix: for the function $u = -x^3 + 3xz + 2y - y^2 -$

$$|H| = \begin{vmatrix} u_{xx} & u_{xy} & u_{xz} \\ u_{yx} & u_{yy} & u_{yz} \\ u_{zx} & u_{zy} & u_{zz} \end{vmatrix}$$

$$u_{xx} = \frac{\partial^2 u}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x}\right) = \frac{\partial}{\partial x}\left(-3x^2 + 3z\right)$$

$$= -3(2x) + 3(0)$$

$$= -6x$$

$$u_{yz} = \frac{\partial^2 u}{\partial y \partial z} = \frac{\partial}{\partial y}\left(\frac{\partial u}{\partial z}\right) = \frac{\partial}{\partial y}(3x - 6z)$$

$$= 0 - 0$$

$$= 0$$

$$u_{xy} = \frac{\partial^2 u}{\partial x \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial u}{\partial y}\right) = \frac{\partial}{\partial x}(2-2y)$$

$$= 0 - 0 = \underline{\underline{0}}$$

$$u_{xz} = \frac{\partial}{\partial x}\left(\frac{\partial u}{\partial z}\right) = \frac{\partial}{\partial x}(3x-6z)$$

$$= 3(1) - 0$$

$$= 3$$

$$u_{yy} = \frac{\partial^2}{\partial y^2} = \frac{\partial}{\partial y}\left(\frac{\partial u}{\partial y}\right) = \frac{\partial}{\partial y}(2-2y) = 0-2(1)$$

$$= -2$$

$$u_{yx} = \frac{\partial}{\partial y}\left(\frac{\partial u}{\partial x}\right) = \frac{\partial}{\partial y}(-3x^2+3z) = 0$$

$$u_{zx} = \frac{\partial^2 u}{\partial z \partial x} = \frac{\partial}{\partial z}\left(\frac{\partial u}{\partial x}\right) = \frac{\partial}{\partial z}(-3x^2+3z)$$

$$= 0 + 3(1)$$

$$= 3$$

$$u_{xz} = \frac{\partial^2 u}{\partial x \partial z} = \frac{\partial}{\partial x}\left(\frac{\partial u}{\partial z}\right) = \frac{\partial}{\partial x}(3x-6z)$$

$$= 3(1) - 0$$

$$= 3$$

$$u_{zz} = \frac{\partial^2 u}{\partial z^2} = \frac{\partial}{\partial z}\left(\frac{\partial u}{\partial z}\right) = \frac{\partial}{\partial z}(3x-6z)$$

$$= 0 - 6(1)$$

$$= -6$$

$\therefore$ Hessian matrix

$$\underline{H} = \begin{pmatrix} -6x & 0 & 3 \\ 0 & -2 & 0 \\ 3 & 0 & -6 \end{pmatrix}$$

At the critical point $(1/2, 1, 1/4)$, the value of

$$\underline{H} = \begin{pmatrix} -6(1/2) & 0 & 3 \\ 0 & -2 & 0 \\ 3 & 0 & -6 \end{pmatrix} = \begin{pmatrix} -3 & 0 & 3 \\ 0 & -2 & 0 \\ 3 & 0 & -6 \end{pmatrix}$$

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$$u_{zy} = \frac{\partial^2 u}{\partial z \, \partial y} = \frac{\partial}{\partial z}\left(\frac{\partial u}{\partial y}\right) = \frac{\partial}{\partial z}(2 - 2y)$$

$$= 0 - 0$$

$$= 0$$

If $|H_1| > 0$, $|H_2| > 0$, $|H_3| = |\bar{H}| > 0$ then the Hessian matrix $\bar{H}$ is said to be positive ~~definition~~ and and it is the condition for minimum.

If $|H_1| < 0$, $|H_2| > 0$, and $|H_3| = |\bar{H}| < 0$ then $\bar{H}$ is said to be negative define and it is the condition for maximum.

$$|H_1| = |-3| = -3 < 0$$

$$|H_2| = \begin{vmatrix} -3 & 0 \\ 0 & -2 \end{vmatrix} = 6 - 0 = 6 > 0$$

$$|H_3| = \begin{vmatrix} -3 & 0 & 3 \\ 0 & -2 & 0 \\ 3 & 0 & -6 \end{vmatrix} = -3(12) - 0(0-0) + 3(0+6)$$

$$= -36 + 18$$

$$= -18 < 0$$

Here, $|H_1| = -3 < 0$, $|H_2| > 0$ and $|H_3| = |\bar{H}| <$

$\therefore$ u satisfies condition for ~~minimum~~ maximum.

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$\therefore$ Maximum value of u is at $(1/2, 1, 1/4)$

$$u = -x^3 + 3xz + 2y - y^2 - 3z^2$$

$$u_{(1/2,1,1/4)} = -\left(\tfrac{1}{2}\right)^3 + 3\left(\tfrac{1}{2}\right)\left(\tfrac{1}{4}\right) + 2(1) - (1)^2 - \cancel{3\left(\tfrac{1}{4}\right)^2} - 3\left(\tfrac{1}{4}\right)^2$$

$$= -\frac{1}{8} + \frac{3}{8} + 2 - 1 - \frac{3}{16}$$

$$= \frac{2}{8} + 1 - \frac{3}{16}$$

$$= \frac{1}{4} + \frac{1}{1} - \frac{3}{16}$$

$$= \frac{4 + 16 - 3}{16}$$

$$= \frac{17}{16}$$

$\therefore$ Minimum value of u is $\boxed{\frac{17}{16}}$